

The University of Chicago

EXTENSIONS OF WARING'S THEOREM
ON SEVENTH POWERS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1938

By

MARGARET EVELYN MAUCH

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1. Introduction

In 1770 E. Waring¹ conjectured that every integer is a sum of at most nine positive integral cubes; also a sum of at most nineteen biquadrates, etc. J. A. Euler stated that, to express every positive integer as a sum of positive m^{th} powers, at least $w = \nu + 2^m - 2$ terms are necessary, where ν is the largest integer $< (3/2)^m$. For $m = 2, 3, 4, 5, 6, 7, 8$, $w = 4, 9, 19, 37, 73, 143, 279$. If it can be proved that a form

$$(a_1, \dots, a_n) = a_1 x_1^m + \dots + a_n x_n^m$$

of weight $w = a_1 + \dots + a_n$ represents every positive integer, Waring's theorem is justified. While no general proof has been made, cubes, biquadrates, fifth and sixth powers have been investigated. L. E. Dickson² has treated the case for nine cubes. All possible theorems on representations by nine cubes are corollaries to

Theorem I. Every positive integer p can be represented by $(11223) = x^3 + y^3 + 2z^3 + 2u^3 + 3v^3$ with $x \geq 0, \dots, v \geq 0$.

All forms of weight nine are given by

¹For a report on the literature, see Dickson's History of the Theory of Numbers, vol. 2, pp. 717-25.

²Extensions of Waring's theorem on nine cubes, Am. Math. Monthly, vol. 34 (1927), pp. 177-83.

Theorem III. If a form of weight nine represents all positive integers, it is one of the following seven: (11223), (111222), (111123), (1111122), (1111113), (11111112), (111111111).

Corresponding results are obtained by Dickson for fourth powers.¹ The principal theorem of the paper is

Theorem 1. If a form of weight 19 represents all positive integers, it is $A = (1122337)$, $B = (1122346)$, $C = (11112238)$, $D = (11112247)$, or $E = (11122228)$, or a form derived from one of the five by partition of the coefficients.

The theorem on fifth powers has also been discussed by Dickson in a paper not yet published. For the case on sixth powers, R. C. Shook² has proved that the minimum order of a form that is to represent all positive integers is ten. He displayed the following 22 forms of order ten that were verified to represent all positive integers up to $4^6 = 4,096$:

(1112457,10,17,25)	(1112457,10,16,26)
(1122457,10,17,24)	(1122457,10,15,26)
(1123457,10,17,23)	(1123457,10,14,26)
(1124457,10,17,22)	(1122457,8,17,26)
(1124557,10,17,21)	(11234577,17,26)
(1124567,10,17,20)	(1122455,10,17,26)
(1124578,10,17,18)	(1123445,10,17,26)
(112457,10,10,16,17)	(1112447,10,17,26)
(112457,10,11,15,17)	(1122347,10,17,26)
(112457,10,12,14,17)	(1112357,10,17,26)
(112457,10,13,13,17)	(1123356,10,21,21).

¹Extensions of Waring's theorem on fourth powers, Bull. Am. Math. Soc., vol. 33 (1927), pp. 319-27.

²An extension of Waring's theorem on sixth powers, University of Chicago, Master's thesis, 1927.

K. C. Yang¹ began a study of forms in seventh powers of weight 143. He proved that the cases with orders ≤ 10 are impossible and displayed one form $f = (11234579, 10, 15, 26, 60)$ of order $n = 12$ that was verified to represent all integers from 1 to $4^7 = 16,384$. In the present paper Yang's results are extended. Further conditions that must be satisfied by any form that is to represent every positive integer are set up. Finally one form $A = (1123469, 13, 17, 41, 46)$ of order 11 was found which was verified to represent all positive integers up to 22,000. The tables employed in this investigation give all decompositions of weight ≤ 143 of integers from 1 to 20,000.

2. Notations and Definitions

The form²

$$(2.1) \quad (a_1, \dots, a_n) \equiv a_1 x_1^7 + \dots + a_n x_n^7,$$

where the $x_i \geq 0$ and $0 < a_1 \leq a_2 \leq \dots$ are all integers, is said to be of order $\underline{n}$ and weight $a_1 + a_2 + \dots + a_n$. Since $ax^7 = x^7 + \dots + x^7$ to $\underline{a}$ terms, a form of weight $\underline{w}$ is equal to a sum of $\underline{w}$ seventh powers. A form (2.1) is said to represent a number p if there exist positive integral or zero solutions x_i of the equation

$$p = a_1 x_1^7 + \dots + a_n x_n^7.$$

¹Various generalizations of Waring's problem, University of Chicago Ph. D. dissertation, 1928, Part III.

²Notations and definitions correspond to those used by Dickson.

But $2175 = 16 \cdot 2^7 + 127$ is not a sum of fewer than 143 seventh powers. Hence the minimum weight of a form (2.1) which represents all positive integers = 143. We seek theorems which imply Waring's theorem as a corollary. Hence we restrict attention to forms (2.1) of weight 143.

Let f be a form (2.1) which represents p and let $a_1 = r + s$. The form $g = (r, s, a_1, \dots, a_n)$ is said to be derived from f by the partition of a_1 into $r + s$. It is evident that any form derived from f by partition represents every integer that can be represented by f and usually further integers.

Write $a = 2^7$, $b = 3^7$, $c = 4^7$, If a positive integer p can be expressed as a linear combination of $1, a, b, \dots$ with integral coefficients ≥ 0 whose sum is ≤ 143 , in one and only one way, p shall be called a simple number. In case there are exactly two such expressions, p is a double number. Similarly, for a triple or k -fold number. For example, $1287 = 10a + 7$ is a simple number and $131 = a + 3$ is a double number.

The notation $(1_s, a_1, \dots, a_n)$ means that the first s coefficients are all unity.

3. Determination of the first eight coefficients $a_1, \dots, a_8$.

In this section we summarize briefly Yang's results. He has proved the following

Theorem. Let s_1 denote $a_1 + a_2 + \dots + a_1$. For $i = 2, 3, \dots, n$, $a_i \leq s_{i-1}$.

Let the form (2.1) represent all positive integers. Then $f = 1$ implies $a_1 = 1$. By the theorem $a_2 = 1$, $a_3 \leq 2$, $a_4 \leq 4$, $a_5 \leq 8$. Since f is to represent all positive integers, certain values of a_6 , a_7 , a_8 are excluded by employing simple numbers. All such excluders¹ are $\leq 16a + 16$ except $b + 6a + 116$, $b + 7a + 109$, $b + 7a + 116$, $b + 13a + 116$, $b + 16a + 110$. For example, the form $(112448, a_7, \dots)$ with $a_7 \geq 11$ is excluded by $10a + 7$ since 10 is not a sum of certain of $(1, 4, 8)$. Also the form $(112335, a_7, \dots)$ with $a_7 \geq 9$ is excluded by $8a + 127$. Thus Yang has constructed a table giving the possible values of $a_1, \dots, a_8$. From the table we have

$$a_6 \leq 9, a_7 \leq 12, a_8 \leq 17, S_8 \equiv a_1 + \dots + a_8 \leq 54.$$

Next the cases $n \leq 10$ are proved to be impossible. For $n = 11$, $32 \leq S_8 \leq 50$. We are interested in the case $n = 11$.

4. Determination of a_n, a_{n-1}, a_{n-2} .

If $a_n > 64$, a partition of f is $(1_{1f}, 65)$, which does not represent the simple number $63 + 16a$. Hence

$$(4.1) \quad a_n \leq 64.$$

If $a_{n-1} > 42$, a partition of f is $(1_{51}, 43, 43)$ which does not represent the simple number $42 + 16a$. Hence

$$a_{n-1} \leq 42.$$

¹On p. 23 of Yang's dissertation, for the forms $(112458, 11, \dots)$ and $(112368, 11, \dots)$, the number $17a + 16 = b + 5$ should be replaced by $110 + 16a + b$.

If $a_{n-2} > 32$, a partition of f is $G = (1_{44}, 33, 33, 33)$. But G does not represent $29 + 16a$. Hence

$$a_{n-2} \leq 32.$$

Forms with $a_{n-1} = 42$, $a_n \geq 45$ are excluded. For, a partition is $(1_{56}, 42, 45)$ which does not represent $41 + 16a$. Similarly, with $a_{n-1} = 41$, $a_n \geq 47$, a partition is $(1_{55}, 41, 47)$ which does not represent $40 + 16a$. Thus employing the simple numbers $41 + 16a$, $40 + 16a$, ..., $32 + 16a$, we have the first ten sets of values of a_n in Table I. The last 32 sets follow from equation (4.1) and because $a_{n-1} \leq a_n$.

TABLE I

a_{n-1}	a_n
42	42-44
41	41-46
40	40-48
- - -	- - - -
33	33-62
32	32-64
31	31-64
- - -	- - - -
2	2-64
1	1-64

Forms with $a_{n-2} = 32$ have $a_{n-1} = a_n = 32$ since the partition $(1_{46}, 32, 32, 33) \neq 31 + 16a$. The following table gives the values of the last three coefficients of f .

TABLE II

a_{n-2}	a_{n-1}	a_n	since	$a_{n-2}+a_{n-1}+a_n$
32	32	32	(1 ₄₄ , 32, 32, 33) $\neq$	31+16a 96
31	31	31-35	(1 ₄₅ , 31, 31, 36) $\neq$	30+16a 93-97
	32	32-34	(1 ₄₅ , 31, 32, 35) $\neq$	" 95-97
	33	33	(1 ₄₅ , 31, 33, 34) $\neq$	" 97
30	30	30-38	(1 ₄₄ , 30, 30, 39) $\neq$	29+16a 90-98
	31	31-37	(1 ₄₄ , 30, 31, 38) $\neq$	" 92-98
	32	32-36	(1 ₄₄ , 30, 32, 37) $\neq$	" 94-98
	33	33-35	(1 ₄₄ , 30, 33, 36) $\neq$	" 96-98
	34	34	(1 ₄₄ , 30, 34, 35) $\neq$	" 98
29	29	29-41	(1 ₄₃ , 29, 29, 42) $\neq$	28+16a 87-99
	30	30-40	(1 ₄₃ , 29, 30, 41) $\neq$	" 89-99
	31	31-39	(1 ₄₃ , 29, 31, 40) $\neq$	" 91-99
	32	32-38	(1 ₄₃ , 29, 32, 39) $\neq$	" 93-99
	33	33-37	(1 ₄₃ , 29, 33, 38) $\neq$	" 95-99
	34	34-36	(1 ₄₃ , 29, 34, 37) $\neq$	" 97-99
	35	35	(1 ₄₃ , 29, 35, 36) $\neq$	" 99
28	28	28-43	(1 ₄₃ , 28, 28, 44) $\neq$	99+16a+2b 84-99
	29	29-42	(1 ₄₃ , 28, 29, 43) $\neq$	" 86-99
	30	30-41	(1 ₄₃ , 28, 30, 42) $\neq$	" 88-99
	31	31-40	(1 ₄₃ , 28, 31, 41) $\neq$	" 90-99
	32	32-39	(1 ₄₃ , 28, 32, 40) $\neq$	" 92-99
	33	33-38	(1 ₄₃ , 28, 33, 39) $\neq$	" 94-99
	34	34-37	(1 ₄₃ , 28, 34, 38) $\neq$	" 96-99
	35	35-36	(1 ₄₃ , 28, 35, 37) $\neq$	" 98-99
27	27	27-45	(1 ₄₃ , 27, 27, 46) $\neq$	99+16a+2b 81-99
	28	28-44	(1 ₄₃ , 27, 28, 45) $\neq$	" 83-99
	29	29-43	(1 ₄₃ , 27, 29, 44) $\neq$	" 85-99
	30	30-42	(1 ₄₃ , 27, 30, 43) $\neq$	" 87-99
	31	31-41	(1 ₄₃ , 27, 31, 42) $\neq$	" 89-99
	32	32-40	(1 ₄₃ , 27, 32, 41) $\neq$	" 91-99
	33	33-39	(1 ₄₃ , 27, 33, 40) $\neq$	" 93-99
	34	34-38	(1 ₄₃ , 27, 34, 39) $\neq$	" 95-99
	35	35-37	(1 ₄₃ , 27, 35, 38) $\neq$	" 97-99
	36	36	(1 ₄₃ , 27, 36, 37) $\neq$	" 99
26	26	26-48	(1 ₄₂ , 26, 26, 49) $\neq$	100+16a+2b 78-100
	27	27-47	(1 ₄₂ , 26, 27, 48) $\neq$	" 80-100
	28	28-46	(1 ₄₂ , 26, 28, 47) $\neq$	" 82-100
	29	29-45	(1 ₄₂ , 26, 29, 46) $\neq$	" 84-100
	30	30-44	(1 ₄₂ , 26, 30, 45) $\neq$	" 86-100
	31	31-43	(1 ₄₂ , 26, 31, 44) $\neq$	" 88-100
	32	32-42	(1 ₄₂ , 26, 32, 43) $\neq$	" 90-100
	33	33-41	(1 ₄₂ , 26, 33, 42) $\neq$	" 92-100
	34	34-40	(1 ₄₂ , 26, 34, 41) $\neq$	" 94-100
	35	35-39	(1 ₄₂ , 26, 35, 40) $\neq$	" 96-100

TABLE II (CONTINUED)

a_{n-2}	a_{n-1}	a_n	since	$a_{n-2}+a_{n-1}+a_n$
26	36	36-38	$(1_{42}, 26, 36, 39) \neq$	100+16a+2b
	37	37	$(1_{42}, 26, 37, 38) \neq$	"
25	25	25-51	$(1_{41}, 25, 25, 52) \neq$	101+16a+2b
	26	26-50	$(1_{41}, 25, 26, 51) \neq$	"
	27	27-49	$(1_{41}, 25, 27, 50) \neq$	"
	28	28-48	$(1_{41}, 25, 28, 49) \neq$	"
	--	--	--	--
	37	37-39	$(1_{41}, 25, 37, 40) \neq$	"
	38	38	$(1_{41}, 25, 38, 39) \neq$	"
24	24	24-54	$(1_{40}, 24, 24, 55) \neq$	102+16a+2b
	25	25-53	$(1_{40}, 24, 25, 54) \neq$	"
	26	26-52	$(1_{40}, 24, 26, 53) \neq$	"
	--	--	--	--
	38	38-40	$(1_{40}, 24, 38, 41) \neq$	"
	39	39	$(1_{40}, 24, 39, 40) \neq$	"
23	23	23-57	$(1_{39}, 23, 23, 58) \neq$	103+16a+2b
	24	24-56	$(1_{39}, 23, 24, 57) \neq$	"
	--	--	--	--
	39	39-41	$(1_{39}, 23, 39, 42) \neq$	"
	40	40	$(1_{39}, 23, 40, 41) \neq$	"
22	22	22-60	$(1_{38}, 22, 22, 61) \neq$	104+16a+2b
	23	23-59	$(1_{38}, 22, 23, 60) \neq$	"
	--	--	--	--
	40	40-42	$(1_{38}, 22, 40, 43) \neq$	"
	41	41	$(1_{38}, 22, 41, 42) \neq$	"
21	21	21-63	$(1_{37}, 21, 21, 64) \neq$	105+16a+b
	22	22-62	$(1_{37}, 21, 22, 63) \neq$	"
	--	--	--	--
	41	41-43	$(1_{37}, 21, 41, 44) \neq$	"
	42	42	$(1_{37}, 21, 42, 43) \neq$	"
20	20	20-64	$a_n \leq 64$	
	21	21-64	"	
	22	22-64	"	
	23	23-64	"	
	24	24-63	$(1_{35}, 20, 24, 64) \neq$	107+16a+b
	25	25-62	$(1_{35}, 20, 25, 63) \neq$	"
	--	--	--	--
	30	30-57	$(1_{35}, 20, 30, 58) \neq$	"
	31	31-56	$(1_{35}, 20, 31, 57) \neq$	"
	32	32-54	$(1_{36}, 20, 32, 55) \neq$	74+14a+4b=
				85+31a+3b
	33	33-52	$(1_{37}, 20, 33, 53) \neq$	72+15a+4b=
				83+32a+3b

TABLE II (CONTINUED)

a_{n-2}	a_{n-1}	a_n	since	$a_{n-2}+a_{n-1}+a_n$	
20	34	34-50	$(1_{38}, 20, 34, 51) \neq$	$70+16a+4b=$ $81+33a+3b$	88-104
	35	35-50	$(1_{37}, 20, 35, 51) \neq$	"	90-105
	36	36-50	$(1_{36}, 20, 36, 51) \neq$	"	92-106
	37	37-50	$(1_{35}, 20, 37, 51) \neq$	$107+16a+b$	94-107
	--	--	--	--	--
	40	40-47	$(1_{35}, 20, 40, 48) \neq$	"	100-107
	41	41-46			102-107
	42	42-44			104-106
19	19	19-64	$a_n \leq 64$		57-102
	20	20-64	"		59-103
	--	--	--	--	--
	25	25-64	"		69-108
	26	26-63	$(1_{34}, 19, 26, 64) \neq$	$108+16a+b$	71-108
	27	27-62	$(1_{34}, 19, 27, 63) \neq$	"	73-108
	--	--	--	--	--
	31	31-58	$(1_{34}, 19, 31, 59) \neq$	"	81-108
	32	32-56	$(1_{35}, 19, 32, 57) \neq$	$75+14a+4b=$ $86+31a+3b$	83-107
	33	33-54	$(1_{36}, 19, 33, 55) \neq$	$73+15a+4b=$ $84+32a+3b$	85-106
	34	34-53	$(1_{36}, 19, 34, 54) \neq$	$71+16a+4b=$ $82+33a+3b$	87-106
	35	35-52	$(1_{36}, 19, 35, 53) \neq$	$71+16a+4b=$ $82+33a+3b$	89-106
	36	36-53	$(1_{34}, 19, 36, 54) \neq$	$108+16a+b$	91-108
	--	--	--	--	--
	39	39-50	$(1_{34}, 19, 39, 51) \neq$	"	97-108
	40	40-48			99-107
	41	41-46			101-106
	42	42-44			103-105
18	18	18-64	$a_n \leq 64$		54-100
	19	19-64	"		56-101
	--	--	--	--	--
	27	27-64	"		72-109
	28	28-63	$(1_{33}, 18, 28, 64) \neq$	$109+16a+b$	74-109
	29	29-62	$(1_{33}, 18, 29, 63) \neq$	"	76-109
	30	30-61	$(1_{33}, 18, 30, 62) \neq$	"	78-109
	31	31-60	$(1_{33}, 18, 31, 61) \neq$	"	80-109
	32	32-58	$(1_{34}, 18, 32, 59) \neq$	$76+14a+4b=$ $87+31a+3b$	82-108
	33	33-56	$(1_{35}, 18, 33, 57) \neq$	$74+15a+4b=$ $85+32+3b$	84-107

TABLE II (CONTINUED)

a_{n-2}	a_{n-1}	a_n	since	$a_{n-2}+a_{n-1}+a_n$		
18	34	34-54	$(1_{36}, 18, 34, 55) \neq$	$72+16a+4b=$ $83+33a+3b$	86-106	
	35	35-53	$(1_{36}, 18, 35, 54) \neq$	$71+16a+4b=$ $82+33a+3b$	88-106	
	36	36-52	$(1_{36}, 18, 36, 53) \neq$	$70+16a+4b=$ $81+33a+3b$	90-106	
	37	37-52	$(1_{35}, 18, 37, 53) \neq$	"	92-107	
	38	38-52			94-108	
	39	39-50			96-107	
	40	40-48			98-106	
	41	41-46			100-105	
	42	42-44			102-104	
	17	17	17-64	$a_n \leq 64$		51-98
		18	18-64	"		53-99
		--	--	--		--
		29	29-64	"		75-110
		30	30-63	$(1_{32}, 17, 30, 64) \neq$	$110+16a+b$	77-110
31		31-62	$(1_{32}, 17, 31, 63) \neq$	"	79-110	
32		32-60	$(1_{33}, 17, 32, 61) \neq$	$77+14a+4b=$ $88+31a+3b$	81-109	
33		33-58	$(1_{34}, 17, 33, 59) \neq$	$75+15a+4b=$ $86+32a+4b$	83-108	
34		34-56	$(1_{35}, 17, 34, 57) \neq$	$73+16a+4b=$ $84+33a+3b$	85-107	
35		35-55	$(1_{35}, 17, 35, 56) \neq$	$72+16a+4b=$ $83+33a+3b$	87-107	
36		36-54	$(1_{35}, 17, 36, 55) \neq$	$71+16a+4b=$ $82+33a+3b$	89-107	
37		37-53	$(1_{35}, 17, 37, 54) \neq$	$70+16a+4b=$ $81+33a+3b$	91-107	
38		38-52			93-107	
39		39-50			95-106	
16	42	42-44			101-103	
	16	16-64	$a_n \leq 64$		48-96	
	17	17-64	"		50-97	
	--	--	--		--	
	30	30-64	"		76-110	
	31	31-64	"		78-111	
	32	32-62	$(1_{32}, 16, 32, 63) \neq$	$78+14a+4b=$ $89+31a+3b$	80-110	
	33	33-60	$(1_{33}, 16, 33, 61) \neq$	$76+15a+4b=$ $87+31a+3b$	82-109	

TABLE II (CONTINUED)

a_{n-2}	a_{n-1}	a_n	since	$a_{n-2}+a_{n-1}+a_n$
16	34	34-59	$(1, 33, 16, 34, 60) \neq$ $75+15a+4b=$ $86+32a+3b$	84-109
	35	35-58		86-109
	36	36-56		88-108
	--	--	--	--
	42	42-44	--	100-102
15	15	15-64	$a_n \leq 64$	45-94
	16	16-64		47-95
	--	--		--
	31	31-64	n	77-110
	32	32-64		79-111
	33	33-62		81-110
	34	34-60	--	83-109
	--	--	--	--
	42	42-44	--	99-101
14	14	14-64	$a_n \leq 64$	42-92
	15	15-64		44-93
	--	--		--
	31	31-64	n	76-109
	32	32-64		78-110
	33	33-62		80-109
	34	34-60	--	82-108
	--	--	--	--
	42	42-44	--	98-100
13	13	13-64	$a_n \leq 64$	39-90
	14	14-64		41-91
	--	--		--
	32	32-64	n	77-109
	33	33-62		79-108
	34	34-60		81-107
	--	--	--	--
	42	42-44	--	97-99
12	12	12-64	$a_n \leq 64$	36-88
	13	13-64		38-89
	14	14-64		40-90
	--	--	--	--
	32	32-64	n	76-108
	33	33-62		78-107
	34	34-60		80-106
	--	--	--	--
	42	42-44	--	96-98
11	11	11-64	$a_n \leq 64$	33-86
	12	12-64		35-87
	--	--		--

TABLE II (CONTINUED)

a_{n-2}	a_{n-1}	a_n	since	$a_{n-2}+a_{n-1}+a_n$
11	31	31-64	$a_n \leq 64$	73-106
	32	32-64		75-107
	33	33-62		77-106
	34	34-60		79-105
	--	--		--
10	42	42-44	$a_{11} \leq 64$	95-97
	10	10-64		30-84
	11	11-64		32-85
	--	--		--
	31	31-64		72-105
	32	32-64		74-106
	33	33-62		76-105
	34	34-60		78-104
9	42	42-44	$a_{11} \leq 64$	94-96
	9	9-64		27-82
	10	10-64		29-83
	--	--		--
	32	32-64		73-105
	33	33-62		75-104
	34	34-60		76-103
	--	--		--
8	42	42-44	$a_{11} \leq 64$	93-95
	--	--		--

5. Cases $S_g = 50, 49$

Theorem 5.1 - No form with $S_g = 50$ represents all integers.

Proof. The only possible set of $a_1, \dots, a_8$ is 112468, 11, 17. The form $(112468, 11, 17, a_9, a_{10}, a_{11})$ with $a_9 \geq 17$ and $a_9 + a_{10} + a_{11} = 93$ does not represent the simple number $115 + 15a + b$. For, $(112468, 11, 17)$ does not represent $22 + 15a + b$ since 15 is not a sum of certain of 2, 6, 8, 11.

Theorem 5.2 - If a form with $S_g = 49$ represents all positive integers, it is one of the following:

(112489, 11, 13, 14, 16, 64)	(112489, 11, 13, 15, 16, 63)
(112489, 11, 13, 14, 17, 63)	(112489, 11, 13, 16, 16, 62)
- - - - -	(112489, 11, 13, 16, 17, 61)
(112489, 11, 13, 14, 25, 55)	- - - - -
(112489, 11, 13, 14, 26, 54)	(112489, 11, 13, 16, 22, 56)
(112489, 11, 13, 15, 15, 64)	(112489, 11, 13, 16, 23, 55)

Proof. The forms $(112489, 11, 13, a_q, a_{10}, a_{11})$ with $a_q = 13$ and $a_q \geq 17$ are excluded by $112 + 16a + b$. Thus $a_q = 14, 15, 16$. The forms $(112489, 11, 13, 14, a_{10}, a_{11})$ with $a_{10} \geq 27$ do not represent $109 + 7a + b$. Similarly, the following forms do not represent the numbers indicated and hence are excluded.

$$\begin{aligned}
 (112489, 11, 13, 15, a_{10} \geq 17, a_{11}) &\not\equiv 110 + 16a + b \\
 (112489, 11, 13, 16, a_{10} \geq 24, a_{11}) &\not\equiv 112 + 7a + b \\
 (112468, 10, 17, a_q, a_{10}, a_{11}) &\not\equiv 110 + 16a + b \\
 (112468, 11, 16, a_q, a_{10}, a_{11}) &\not\equiv 115 + 15a + b \\
 (112458, 11, 17, a_q, a_{10}, a_{11}) &\not\equiv 110 + 16a + b \\
 (112368, 11, 17, a_q, a_{10}, a_{11}) &\not\equiv 110 + 16a + b.
 \end{aligned}$$

6. Case $S_g = 32$

From Table II the values of a_q, a_{10}, a_{11} are 15, 32, 64 and 16, 31, 64.

The following numbers are used as excluders:

- (1) $45 + 36a = 34 + 19a + b = 23 + 2a + 2b$
- (2) $46 + 36a = 35 + 19a + b = 24 + 2a + 2b$
- (3) $105 + 10a + 2b$
- (4) $98 + 19a + 2b = 87 + 2a + 3b$
- (5) $53 + 54a = 42 + 37a + b = 31 + 20a + 2b = 20 + 3a + 3b$
- (6) $95 + 20a + 2b = 84 + 3a + 3b$
- (7) $97 + 20a + 2b = 86 + 3a + 3b$
- (8) $98 + 20a + 2b = 87 + 3a + 3b$
- (9) $100 + 20a + 2b = 89 + 3a + 3b$
- (10) $52 + 55a = 41 + 38a + b = 30 + 21a + 2b = 19 + 4a + 3b$

- (11) $94 + 21a + 2b = 83 + 4a + 3b$
- (12) $93 + 21a + 3b = 82 + 4a + 4b$
- (13) $92 + 22a + 3b = 81 + 5a + 4b$
- (14) $94 + 22a + 3b = 83 + 5a + 4b$
- (15) $90 + 24a + 3b = 79 + 7a + 4b$
- (16) $59 + 56a + 2b = 48 + 39a + 3b = 37 + 22a + 4b = 26 + 5a + 5b$
- (17) $61 + 56a + 2b = 50 + 39a + 3b = 39 + 22a + 4b = 28 + 5a + 5b$
- (18) $82 + 31a + 4b = 71 + 14a + 5b$
- (19) $83 + 31a + 4b = 72 + 14a + 5b$
- (20) $59 + 56a + 3b = 48 + 39a + 4b = 37 + 22a + 5b = 26 + 5a + 6b$
- (21) $60 + 56a + 3b = 49 + 39a + 4b = 38 + 22a + 5b = 27 + 5a + 6b$
- (22) $59 + 57a + 3b = 48 + 40a + 4b = 37 + 23a + 5b = 26 + 6a + 6b$
- (23) $60 + 57a + 3b = 49 + 40a + 4b = 38 + 23a + 5b = 27 + 6a + 6b$
- (24) $74 + 57a + 4b = 63 + 40a + 5b = 52 + 23a + 6b = 41 + 6a + 7b$

The following forms are excluded:

- | | |
|---|---|
| (1 ₂ , 11, 15, 32, 64) by (25) | (1 ₁₆ , 8, 8, 15, 32, 64) by (19) |
| (1 ₂₁ , 11, 16, 31, 64) by (3) | (1 ₁₄ , 8, 10, 16, 31, 64) by (16) |
| (1 ₁₄ , 8, 10, 15, 32, 64) by (19) | (1 ₁₅ , 8, 9, 16, 31, 64) by (16) |
| (1 ₁₅ , 8, 9, 15, 32, 64) by (19) | (1 ₁₅ , 8, 8, 16, 31, 64) by (16). |

This proves that $a_1 \leq 10$ and when $a_1 = 8, 9$, or 10 , $a_2 \leq 7$.

All forms

- $$\begin{array}{ll} (111148, a_1, \dots, a_n) & (111146, a_1, \dots, a_n) \\ (111147, a_1, \dots, a_n) & (111137, a_1, \dots, a_n) \end{array}$$

are excluded by (14).

Let $a_g = 10$. The following forms are excluded by the numbers indicated:

- (1113466,10,15,32,64) by (20) (1124455,10,15,32,64) by (7)
 (1113466,10,16,31,64) by (1) (1124455,10,16,31,64) by (4)
 (1113367,10,15,32,64) by (20) (1124446,10,15,32,64) by (7)
 (1112467,10,15,32,64) by (7) (1124446,10,15,32,64) by (4)
 (1112377,10,a₁,a₁₀,a₁₁) by (15) (1123366,10,15,32,64) by (18)
 (1122277,10,a₁,a₁₀,a₁₁) by (17).

Theorem 6.1 - The following forms are the only forms

with $S_0 = 32$, $a_0 = 10$ which can represent all positive integers.

- | | |
|-----------------------|-----------------------|
| (1113457,10,15,32,64) | (1123456,10,15,32,64) |
| (1113457,10,16,31,64) | (1123456,10,16,31,64) |
| (1113367,10,16,31,64) | (1123447,10,15,32,64) |
| (1112467,10,16,31,64) | (1123447,10,16,31,64) |

(1123366, 10, 16, 31, 64)	(1122466, 10, 16, 31, 64)
(1123357, 10, 15, 32, 64)	(1122457, 10, 15, 32, 64)
(1123357, 10, 16, 31, 64)	(1122457, 10, 16, 31, 64)
(1122466, 10, 15, 32, 64)	(1122367, 10, 15, 32, 64)
	(1122367, 10, 16, 32, 64)

Let $a_g = 9$. The following forms are excluded:

(11135669, 15, 32, 64) by (1)	(11244569, 15, 32, 64) by (7)
(11135669, 16, 31, 64) by (2)	(11244569, 16, 31, 64) by (4)
(11135579, 15, 32, 64) by (9)	(11244479, 15, 32, 64) by (7)
(11135579, 16, 32, 64) by (8)	(11244479, 16, 31, 64) by (4)
(11134679, 15, 32, 64) by (9)	(11235569, 15, 32, 64) by (13)
(11133779, a_q, a_{10}, a_{11}) by (21)	(11235569, 16, 31, 64) by (13)
(11125679, 15, 32, 64) by (6)	(11234669, 15, 32, 64) by (20)
(11125679, 16, 31, 64) by (11)	(11233679, 15, 32, 64) by (20)
(11124779, 15, 32, 64) by (6)	(11225669, a_q, a_{10}, a_{11}) by (12)
(11124779, 16, 31, 64) by (16)	(11225579, a_q, a_{10}, a_{11}) by (12)
(11245559, 15, 32, 64) by (7)	(11224679, 15, 32, 64) by (20)
(11245559, 16, 31, 64) by (4)	(11223779, a_q, a_{10}, a_{11}) by (13).

Theorem 6.2 - If a form with $S_g = 32$, $a_g = 9$ represents all positive integers, it is one of the following six:

(11134679, 16, 31, 64)	(11234579, 16, 31, 64)
(11234669, 16, 31, 64)	(11233679, 16, 31, 64)
(11234579, 15, 32, 64)	(11224679, 16, 31, 64).

Let $a_g = 8$. The following forms are excluded:

(11136668, a_q, a_{10}, a_{11}) by (13)	(11244578, 15, 32, 64) by (7)
(11135678, a_q, a_{10}, a_{11}) by (13)	(11244578, 16, 31, 64) by (4)
(11134778, 15, 32, 64) by (22)	(11235668, a_q, a_{10}, a_{11}) by (13)
(11134778, 16, 31, 64) by (21)	(11235578, a_q, a_{10}, a_{11}) by (13)
(11125778, 15, 32, 64) by (6)	(11233778, 15, 32, 64) by (24)
(11125778, 16, 31, 64) by (11)	(11233778, 16, 31, 64) by (23)
(11245568, 15, 32, 64) by (7)	(11226668, a_q, a_{10}, a_{11}) by (12)
(11245568, 16, 31, 64) by (4)	(11225678, a_q, a_{10}, a_{11}) by (12)
(11244668, 15, 32, 64) by (7)	(11224778, 15, 32, 64) by (7)
(11244668, 16, 31, 64) by (4)	(11224778, 16, 31, 64) by (10).

Theorem 6.3 - The only forms with $S_g = 32$, $a_g = 8$ which can represent all positive integers are the following:

(11234678, 15, 32, 64)	(11234678, 16, 31, 64).
------------------------	-------------------------

Let $a_g = 6, 7$. The following forms are excluded:

(11136677, a_q, a_{10}, a_{11}) by (13)	(11244677, 15, 32, 64) by (7)
(11135777, a_q, a_{10}, a_{11}) by (13)	(11244677, 16, 31, 64) by (9)
(11246666, 15, 32, 64) by (7)	(11236667, a_q, a_{10}, a_{11}) by (13)
(11246666, 16, 31, 64) by (15)	(11235677, a_q, a_{10}, a_{11}) by (13)
(11245667, 15, 32, 64) by (7)	(11234777, 15, 32, 64) by (24)
(11245667, 16, 31, 64) by (9)	(11234777, 16, 31, 64) by (21)
(11245577, 15, 32, 64) by (7)	(11226677, a_q, a_{10}, a_{11}) by (12)
(11245577, 16, 31, 64) by (9)	(11225777, a_q, a_{10}, a_{11}) by (12)

Theorem 6.4 - No form with $S_g = 32$, $a_g \leq 7$ represents all integers.

This concludes the case $S_g = 32$.

7. Case $S_g = 33$

From Table II there are ten sets of values for a_q , a_{10} , a_{11} . The following additional numbers are employed as excluders:

$$\begin{aligned}
 47 + 36a &= 36 + 19a + b = 25 + 2a + 2b \\
 97 + 19a + b &= 86 + 2a + 2b \\
 99 + 19a + b &= 88 + 2a + 2b \\
 100 + 19a + b &= 89 + 2a + 2b \\
 108 + 19a + b &= 97 + 2a + 2b \\
 92 + 24a + b &= 81 + 7a + 2b \\
 96 + 20a + 2b &= 85 + 3a + 3b \\
 97 + 21a + 2b &= 86 + 4a + 3b \\
 91 + 23a + 2b &= 80 + 6a + 3b \\
 92 + 23a + 2b &= 81 + 6a + 3b \\
 91 + 24a + 2b &= 80 + 7a + 3b \\
 61 + 55a + b &= 50 + 38a + 2b = 39 + 21a + 3b = 28 + 4a + 4b \\
 62 + 55a + b &= 51 + 38a + 2b = 40 + 21a + 3b = 29 + 4a + 4b \\
 94 + 21a + 3b &= 83 + 4a + 4b \\
 93 + 22a + 3b &= 82 + 5a + 4b \\
 91 + 23a + 3b &= 80 + 6a + 4b \\
 92 + 23a + 3b &= 81 + 6a + 4b \\
 93 + 23a + 3b &= 82 + 6a + 4b \\
 91 + 24a + 3b &= 80 + 7a + 4b \\
 92 + 24a + 3b &= 81 + 7a + 4b \\
 93 + 24a + 3b &= 82 + 7a + 4b \\
 62 + 56a + 2b &= 51 + 39a + 3b = 40 + 22a + 4b = 29 + 5a + 5b \\
 60 + 59a + 4b &= 49 + 42a + 5b = 38 + 25a + 6b = 27 + 8a + 7b.
 \end{aligned}$$

By proceeding as in sections 5 and 6, we obtain the

following theorems for $S_g = 33$. All forms not listed are excluded.

Theorem 7.1 - For $S_g = 33$, $a_1 = 8$, $a_g = 11$, the only forms that can represent all positive integers are the following:

(1112458, 11, 15, 31, 63)	(1123348, 11, 17, 31, 62)
(1112458, 11, 15, 32, 62)	(1122448, 11, 15, a_{10} , a_{11})
(1123348, 11, 15, a_{10} , a_{11})	(1122358, 11, 15, a_{10} , a_{11})
(1123348, 11, 16, 31, 63)	(1122358, 11, 16, 31, 63)
(1123348, 11, 16, 32, 62)	(1122358, 11, 16, 32, 62)
	(1122358, 11, 17, 31, 62)

Theorem 7.2 - For $S_g = 33$, $a_1 = 7$, $a_g = 11$, the only forms that can represent all positive integers are the following:

(1113457, 11, 14, a_{10} , a_{11})	(1123357, 11, 14, a_{10} , a_{11})
(" , 15, a_{10} , a_{11})	(" , 15, a_{10} , a_{11})
(" , 16, a_{10} , a_{11})	(" , 16, a_{10} , a_{11})
(" , 17, 30, 63)	(" , 17, 30, 63)
(" , 17, 31, 62)	(" , 17, 31, 62)
(1113367, 11, 16, a_{10} , a_{11})	(1122457, 11, 14, a_{10} , a_{11})
(" , 17, 30, 63)	(" , 15, a_{10} , a_{11})
(" , 17, 31, 62)	(" , 16, a_{10} , a_{11})
(1112467, 11, 15, a_{10} , a_{11})	(" , 17, 30, 63)
(" , 16, a_{10} , a_{11})	(" , 17, 31, 62)
(" , 17, 30, 63)	(1122367, 11, 14, a_{10} , a_{11})
(" , 17, 31, 62)	(" , 15, a_{10} , a_{11})
(1123447, 11, 14, a_{10} , a_{11})	(" , 16, a_{10} , a_{11})
(1123447, 11, 15, a_{10} , a_{11})	(" , 17, 30, 63)
(" , 16, a_{10} , a_{11})	(" , 17, 31, 62).
(" , 17, 30, 63)	
(" , 17, 31, 62)	

Theorem 7.3 - If a form having $S_g = 33$, $a_1 = 6$, $a_g = 11$ represents all positive integers, it is one of the following:

(1113466, 11, 15, a_{10}, a_{11})	(1123366, 11, 15, a_{10}, a_{11})
(" , 16, a_{10}, a_{11})	(" , 16, a_{10}, a_{11})
(1123456, 11, 14, a_{10}, a_{11})	(" , 17, 30, 63)
(" , 15, a_{10}, a_{11})	(" , 17, 31, 62)
(" , 16, a_{10}, a_{11})	(1122466, 11, 14, a_{10}, a_{11})
(" , 17, 30, 63)	(" , 15, a_{10}, a_{11})
(" , 17, 31, 62)	(" , 16, a_{10}, a_{11})
(1123366, 11, 14, a_{10}, a_{11})	(" , 17, 30, 63)
	(" , 17, 31, 62).

Theorem 7.4 - No form with $S_g = 33$, $a_1 \leq 5$, $a_g = 11$ represents all integers.

Theorem 7.5 - For $S_g = 33$, $a_g = 10$, the only forms that can represent all integers are the following:

(1113467, 10, 14, a_{10}, a_{11})	(1124456, 10, 15, a_{10}, a_{11})
(" , 15, a_{10}, a_{11})	(" , 16, a_{10}, a_{11})
(" , 16, a_{10}, a_{11})	(" , 17, a_{10}, a_{11})
(1113458, 10, 14, a_{10}, a_{11})	(1124447, 10, 15, a_{10}, a_{11})
(" , 15, a_{10}, a_{11})	(" , 16, a_{10}, a_{11})
(" , 16, 31, 63)	(" , 17, a_{10}, a_{11})
(" , 16, 32, 62)	(1123556, 10, 17, a_{10}, a_{11})
(" , 17, 31, 62)	(1123466, 10, a_q, a_{10}, a_{11})
(1113368, 10, 14, a_{10}, a_{11})	(1123457, 10, a_q, a_{10}, a_{11})
(" , 15, a_{10}, a_{11})	(1123448, 10, a_q, a_{10}, a_{11})
(" , 16, 31, 63)	(1123367, 10, a_q, a_{10}, a_{11})
(" , 16, 32, 62)	(1123358, 10, a_q, a_{10}, a_{11})
(1112567, 10, 17, a_{10}, a_{11})	(1122566, 10, 17, a_{10}, a_{11})
(1112477, 10, 15, a_{10}, a_{11})	(1122467, 10, 16, 30, 64)
(" , 16, a_{10}, a_{11})	(" , 17, 29, 64)
(" , 17, a_{10}, a_{11})	(" , 17, 30, 63)
(1112468, 10, 15, a_{10}, a_{11})	(1122458, 10, 15, a_{10}, a_{11})
(" , 16, a_{10}, a_{11})	(" , 16, a_{10}, a_{11})
(" , 17, a_{10}, a_{11})	(" , 17, a_{10}, a_{11})
(1124555, 10, 15, a_{10}, a_{11})	
(" , 16, a_{10}, a_{11})	
(" , 17, a_{10}, a_{11})	

Theorem 7.5 - For $S_g = 33$, $a_g = 9$, the only forms that can represent all integers are the following:

(11135679, 17, a_{10}, a_{11})	(11125779, 17, a_{10}, a_{11})
(11134779, 14, a_{10}, a_{11})	(11125689, 17, a_{10}, a_{11})
(" , 15, a_{10}, a_{11})	(11245569, 15, a_{10}, a_{11})
(" , 16, a_{10}, a_{11})	(" , 16, a_{10}, a_{11})
(11134689, a_q, a_{10}, a_{11})	(" , 17, a_{10}, a_{11})

(11244669, 15, a_{10}, a_{11})	(11233689, 14, a_{10}, a_{11})
(" , 16, a_{10}, a_{11})	(" , 15, 32, 63)
(" , 17, a_{10}, a_{11})	(" , 15, 33, 62)
(11244579, 15, a_{10}, a_{11})	(" , 16, 31, 63)
(" , 16, a_{10}, a_{11})	(" , 16, 32, 62)
(" , 17, a_{10}, a_{11})	(" , 17, 30, 63)
(11235669, 17, a_{10}, a_{11})	(" , 17, 31, 62)
(11235579, 17, a_{10}, a_{11})	(11225679, 17, a_{10}, a_{11})
(11234679, 16, 30, 64)	(11224779, 17, 29, 64)
(" , 17, a_{10}, a_{11})	(11224689, 15, 33, 62)
(11234589, a_4, a_{10}, a_{11})	(" , 16, 30, 64)
(11233779, 15, 33, 62)	(" , 16, 32, 62)
(" , 16, 32, 62)	(" , 17, a_{10}, a_{11})
(" , 17, a_{10}, a_{11})	

Theorem 7.6 - If a form having $S_g = 33$, $a_g = 8$ represents all positive integers, it is one of the following:

(11135778, 17, a_{10}, a_{11})	(11234778, 16, 30, 64)
(11134788, 14, a_{10}, a_{11})	(" , 17, a_{10}, a_{11})
(" , 17, 30, 63)	(11234688, 16, 30, 64)
(" , 17, 31, 62)	(" , 17, a_{10}, a_{11})
(11244588, 17, 30, 63)	(11233788, 17, 29, 64)
(" , 17, 31, 62)	(11225778, 17, a_{10}, a_{11})
(11235678, 17, a_{10}, a_{11})	(11225688, 17, 30, 63)
(11235588, 17, 30, 63)	(" , 17, 31, 62)
(" , 17, 31, 62)	(11224788, 17, 30, 63)

Theorem 7.7 - No form with $S_g = 33$, $a_q \leq 7$ represents all integers.

This concludes the case $S_g = 33$.

8. Certain Forms of Orders 11 and 12

The form $A = (1123469, 13, 17, 40, 47)$ was verified to represent all the numbers from 1 to 20,000 except

$$10,359 = 75 + 12a + 4b = 86 + 29a + 3b.$$

Of the sixty-eight partitions of A into forms of order 12, the following only do not represent 10,359:

(11113469,13,17,40,47)	(11222369,13,17,40,47)
(11122469,13,17,40,47)	(11123468,13,17,40,47) (8.1)
(11123369,13,17,40,47)	(11223467,13,17,40,47).

Hence

Theorem 8.1 - The sixty-two partitions of $A = (1123469,13,17,40,47)$ of order 12 not listed in (8.1) are the only partitions of A of order 12 that can possibly represent all integers.

The form $B = (1123469,13,17,41,46)$ of order 11 was verified to represent all integers from 1 to 22,000.

Theorem 8.2 - The form $B = (1123469,13,17,41,46)$ of order 11 represents all positive integers $\leq 22,000$.

VITA

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